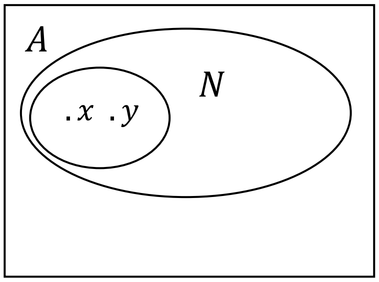
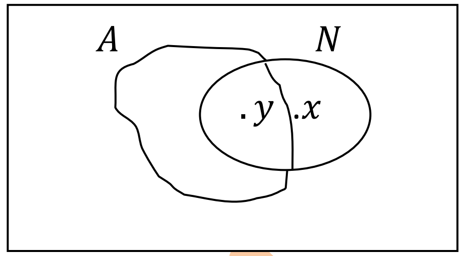
Limit point of a set

Let (X, T) be a topological space and $A \subset X$ then a point $x \in X$ is called a limit point (cluster or accumulation point) of set A **iff every nbd** of x contains a point of A other than x.





Derived set –The set of all limit points of a set A is called Derived set of A and is denoted by $D(A)$.

Example 1: Let $X = \{a, b, c\}$ and $T = \{\emptyset, \{a\}, \{a, b\}, \{a, c\}, X\}$ find all limit points of the sets

(i) $A = \{b, c\}$ (ii) $B = \{a, c\}$

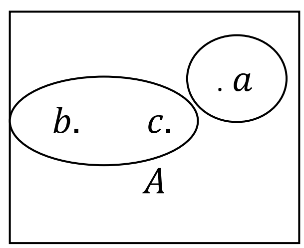
Solution – (i) **Test for limit points of set $A = \{b, c\}$**:

For point a:

T –nbds of point a are: $\{a\}, \{a, b\}, \{a, c\}, X$

In which $\{a\}$ contains no points of $A = \{b, c\}$ other than a

$\therefore$ a is **not a limit point** of set A.



For point b:

T –nbds of point b are: $\{a, b\}, X$

In which $\{a, b\}$ contains no points of $A = \{b, c\}$ other than b

$\therefore$ b is **not a limit point** of set A.

For point c:

T –nbds of point c are: $\{a, c\}, X$

In which $\{a, c\}$ contains no points of $A = \{b, c\}$ other than c

$\therefore$ c is **not a limit point** of set A.

(ii) **Test for limit points of set $\boldsymbol{B} = \{a, c\}$:**

<u>For point</u> a:

T –nbds of point a are: $\{a\}, \{a, b\}, \{a, c\}, X$

In which $\{a\}$ contains no points of $B = \{a, c\}$ other than a

$\therefore a$ is **not a limit point** of set A.

<u>For point b</u>:

T –nbds of point b are: $\{a, b\}, X$

In which $\{a, b\}$ contains point a of $B = \{a, c\}$ other than b

Also X contains points a and c of $B = \{a, c\}$ other than b

$\therefore b$ is **a limit point** of set B.

<u>For point c</u>:

T –nbds of point c are: $\{a, c\}, X$

In which $\{a, c\}$ contains point a of $B = \{a, c\}$ other than c

And X contains points a of $B = \{a, c\}$ other than c

$\therefore c$ is **a limit point** of set B.

Hence $D(A) = \emptyset$ and $D(B) = \{b, c\}$

Example 2.Let $X = \{a, b, c, d, e\}$ and $T = \{\emptyset, \{b\}, \{d, e\}, \{b, d, e\}, \{a, c, d, e\}, X\}$

find all limit points of the set $A = \{b, c, d\}$

Solution- **<u>For point</u> $\boldsymbol{a}$:**

T –nbds of point a are: $\{a, c, d, e\}, X$

In which $\{a, c, d, e\}$ contains points c, d of $A = \{b, c, d\}$ other than a

And X contains points b, c and d of $A = \{b, c, d\}$ other than a

Thus every nbd of point a contains a point of A other than a.

$\therefore a$ is **a limit point** of set A.

For point b:

T −nbds of point b are: $\{b\}, \{b, d, e\}, X$

In which $\{b\}$ contains no point of a of $A = \{b, c, d\}$ other than b

$\therefore b$ is **a not a limit point** of set A.

For point c:

T −nbds of point c are: $\{a, c, d, e\}, X$

In which $\{a, c, d, e\}$ contains point d of $A = \{b, c, d\}$ other than c

and X contains points b, d of $A = \{b, c, d\}$ other than c

Thus every nbd of point c contains a point of A other than c

$\therefore c$ is **a limit point** of set A.

For point d:

T −nbds of point d are: $\{d, e\}, \{b, d, e\}, \{a, c, d, e\}, X$

In which $\{d, e\}$ contains no points of $A = \{b, c, d\}$ other than d

$\therefore d$ is **a limit point** of set A.

For point e:

T −nbds of point e are: $\{d, e\}, \{b, d, e\}, \{a, c, d, e\}, X$

In which $\{d, e\}$ contains point d of $A = \{b, c, d\}$ other than e

$\{b, d, e\}$ contains points b and d of $A = \{b, c, d\}$ other than e

$\{a, c, d, e\}$ contains points c and d of $A = \{b, c, d\}$ other than e

X contains point b, c and d of $A = \{b, c, d\}$ other than e

$\therefore e$ is **a limit point** of set A.

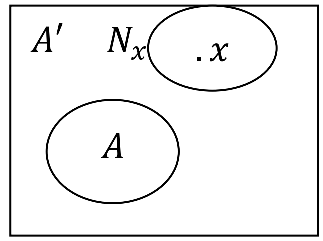
Hence $D(A) = \{a, c, e\}$

Theorem –Let (X, T) be a topological space and let A be subset of X.Then A is closed if and only if $D(A) \subset A$ i.e. iff $A \cup D(A) = A$

Proof – Let A be T –closed then

A is T –closed $\Rightarrow A'$ is T –open

$\Rightarrow$ to each $x \in A'$, $\exists$ a nbd N_x of x s.t. $N_x \subset A'$



Since A and A' are disjoint $\Rightarrow$ A and N_x are disjoint

$\Rightarrow A \cap N_x = \emptyset$

$\Rightarrow$ nbd of x contains no point of A

$\Rightarrow x$ is not a limit point of A

Thus $x \in A' \Rightarrow x$ is not a limit point of A

i.e. no point of A' is a limit point of A which implies that set A contains all its limit points. Hence $D(A) \subset A$

conversely, let $D(A) \subset A$ and $x \in A'$ then

$x \in A'$

$\Rightarrow x \notin A$

$\Rightarrow x \notin D(A)$ [since $D(A) \subset A$]

$\Rightarrow x$ is not a limit point of A

$\Rightarrow \exists$ a nbd of x which does not contain any point of A

$\Rightarrow \exists$ a nbd N_x of x such that $N_x \cap A = \emptyset$,

$\Rightarrow x \in N_x \subset A'$

$\Rightarrow$ A' is a nbd of x

$\Rightarrow$ A' is a nbd of each of its point

$\Rightarrow A'$ is open

$\Rightarrow A$ is closed

Properties of derived sets

Theorem –Let (X,T) be a topological space and A *and* B be subsets of a topological space. Then

(i) $D(\emptyset)=\emptyset$

(ii) $A\subset B\Rightarrow D(A)\subset D(B)$

(iii) $D(A\cap B)\subset D(A)\cap D(B)$

(iv) $D(A\cup B)=D(A)\cup D(B)$

Proof : (i) since $\emptyset$ is closed $\Rightarrow D(\emptyset)\subset\emptyset$ [A is closed $\Leftrightarrow D(A)\subset A$]

But $\emptyset\subset D(\emptyset)$ [$\emptyset$ is a subset of every set]

Hence $D(\emptyset)=\emptyset$

(ii) Let $x\in D(A)$ then

x is a limit point of set A

$\Rightarrow$ Every nbd of x must contain a point of set A different from x.

$\Rightarrow$ Every nbd of x must contain a point of set B different from x. [$\because A\subset B$]

$\Rightarrow$ x is a limit point of set B

$\Rightarrow$ $x\in D(B)$

Thus we have $x\in D(A)\Rightarrow x\in D(B)$

$\therefore \quad D(A)\subset D(B)$

Hence $A\subset B\Rightarrow D(A)\subset D(B)$

(iii) Since $A\cap B\subset A$ *and* $A\cap B\subset B$

$A\cap B\subset A\Rightarrow D(A\cap B)\subset D(A)$

And $A\cap B\subset B\Rightarrow D(A\cap B)\subset D(B)$ [$\because A\subset B\Rightarrow D(A)\subset D(B)$]

Consequently $D(A\cap B)\subset D(A)\cap D(B)$

(iv) $\because$ *we know that*, $A\subset A\cup B$ and $B\subset A\cup B$

Then $A\subset A\cup B\Rightarrow D(A)\subset D(A\cup B)$

And $B\subset A\cup B\Rightarrow D(B)\subset D(A\cup B)$ [$\because A\subset B\Rightarrow D(A)\subset D(B)$]

$\therefore\ D(A) \cup D(B) \subset D(A \cup B)$ ....(i)

Now we shall prove that

$$D(A \cup B) \subset D(A) \cup D(B)$$

i.e. $x \in D(A \cup B) \Rightarrow x \in D(A) \cup D(B)$

it can be proved by proving its contra-positive statement:

$$x \notin D(A) \cup D(B) \Rightarrow x \notin D(A \cup B)$$

Now $x \notin D(A) \cup D(B) \Rightarrow x \notin D(A)$ and $x \notin D(B)$

$\Rightarrow x$ is not a limit point of A and x is not a limit point of B

$\Rightarrow \exists$ a nbd N_x of x such that $(N_x - \{x\}) \cap A = \emptyset$ and $(N_x - \{x\}) \cap B = \emptyset$

$\Rightarrow [(N_x - \{x\}) \cap A] \cup [(N_x - \{x\}) \cap B] = \emptyset$

$\Rightarrow (N_x - \{x\}) \cap (A \cup B) = \emptyset$

$\Rightarrow x$ is not a limit point of $A \cup B$

$\Rightarrow x \notin D(A \cup B)$

Thus we have proved that $x \notin D(A) \cup D(B) \Rightarrow x \notin D(A \cup B)$

So its contra-positive $x \in D(A \cup B) \Rightarrow x \in D(A) \cup D(B)$ is true.

$\Rightarrow D(A \cup B) \subset D(A) \cup D(B)$ ....(ii)

Hence from (i) and (ii) we have

$$D(A \cup B) = D(A) \cup D(B)$$

Proved