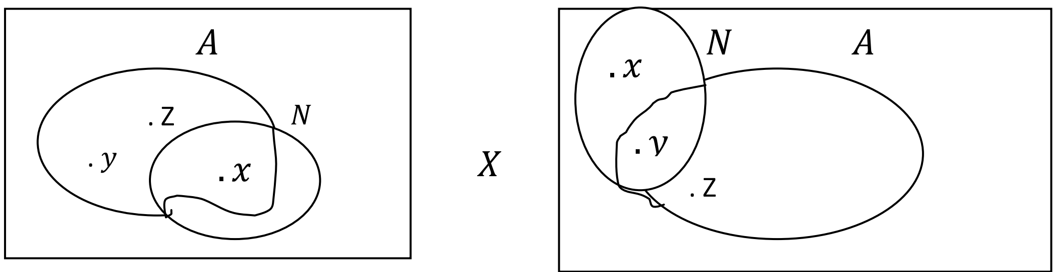
Adherent point

Let (X, T) be a topological space and $A \subset X$ then a point $x \in X$ is called an adherent point of set A **iff every nbd** of x contains at least one point of A.



i.e. $N \cap A \neq \emptyset$

Example- Let $X = \{a, b, c, d, e\}$ and $T = \{\emptyset, \{b\}, \{d, e\}, \{b, d, e\}, \{a, c, d, e\}, X\}$

find all adherent points of the set $A = \{b, c, d\}$

Solution- **<u>For point</u> $\boldsymbol{a}$:**

T −nbds of point a are: $\{a, c, d, e\}, X$

In which $\{a, c, d, e\}$ contains points c, d of $A = \{b, c, d\}$

And X contains points b, c *and* d of $A = \{b, c, d\}$

Thus every nbd of point a contains at least one point of A

$\therefore$ a is **an adherent point** of set A.

<u>For point</u> $\boldsymbol{b}$:

T −nbds of point b are: $\{b\}, \{b, d, e\}, X$

In which $\{b\}$ contains point of b of $A = \{b, c, d\}$

$\{b, d, e\}$ contains points b and d of $A = \{b, c, d\}$

X contains points b, c and d of $A = \{b, c, d\}$

Thus every nbd of point b contains at least one point of A

$\therefore b$ is **an adherent point** of set A.

For point c:

T −nbds of point c are: $\{a,c,d,e\}, X$

In which $\{a,c,d,e\}$ contains points c and d of $A=\{b,c,d\}$

and X contains points b, c and d of $A=\{b,c,d\}$

Thus every nbd of point c contains at least one point of A

$\therefore c$ is **an adherent point** of set A.

For point d:

T −nbds of point d are: $\{d,e\},\{b,d,e\},\{a,c,d,e\}, X$

In which $\{d,e\}$ contains point d of $A=\{b,c,d\}$

$\{b,d,e\}$ contains points b and d of $A=\{b,c,d\}$

$\{a,c,d,e\}$ contains points c and d of $A=\{b,c,d\}$

and X contains points b, c and d of $A=\{b,c,d\}$

Thus every nbd of point d contains at least one point of A

$\therefore d$ is **an adherent point** of set A.

For point e:

T −nbds of point e are: $\{d,e\},\{b,d,e\},\{a,c,d,e\}, X$

In which $\{d,e\}$ contains point d of $A=\{b,c,d\}$

$\{b,d,e\}$ contains points b and d of $A=\{b,c,d\}$

$\{a,c,d,e\}$ contains points c and d of $A=\{b,c,d\}$

X contains points b, c and d of $A=\{b,c,d\}$

$\therefore e$ is **an adherent point** of set A.

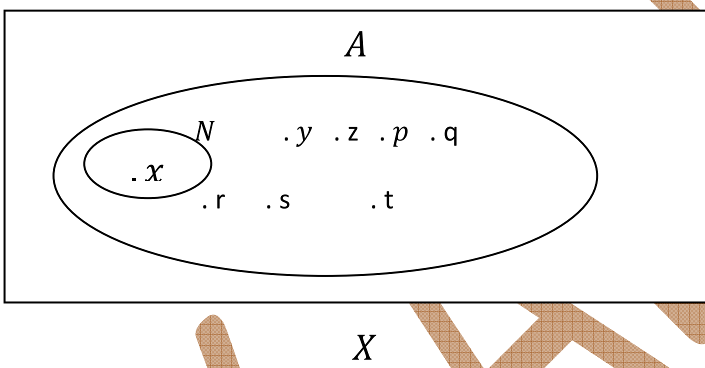
Isolated point

Let (X,T) be a topological space and $A \subset X$ then a point $x \in X$ is called an isolated point of set A **iff**

(i) $x \in A$

(ii) There exists at least one nbd N_x of x which does not contain any other point of A (i.e. x is not a limit point of set A)

i.e $$(N_x - \{x\}) \cap A = \emptyset$$



Condensation Point

Let (X,T) be a topological space and $A \subset X$ then a point $x \in X$ is called condensation point of A if every nbd of x contains uncountably many points of A.

Consequently every condensation point of set A is a limit point of A.

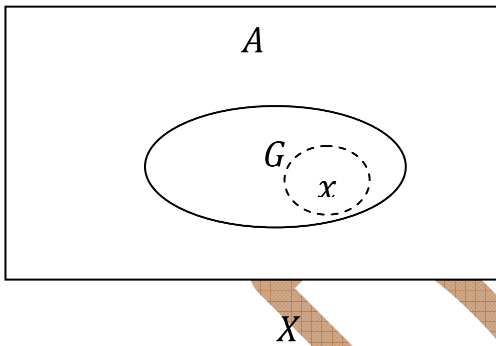
Note –(i) Every limit point is an adherent point of set A but converse is not true.

(ii) Every condensation point of set A is a limit point of A but converse is not true.

Interior points

Let (X, T) be a topological space and $A \subset X$ then a point $x \in X$ is said to be an interior point of set A iff A is a nbd of x i.e iff there exists an open set G such that

$$x \in G \subset A$$



The set of all interior points of set A is denoted by A°, A^{i} or $Int(A)$ or $i(A)$.

Note - $A^{\circ} \subset A$

$x \in A^{\circ} \Rightarrow x$ is an interior point of set A

$\Rightarrow$ A is a nbd of x

$\Rightarrow x \in A$

Thus $x \in A^{\circ} \Rightarrow x \in A$

Hence $A^{\circ} \subset A$.

Theorem - $A^{\circ} = \cup \{G \subset A : G \text{ } is \text{ } open\}$

Proof - $x \in A^{\circ} \Leftrightarrow x$ is an interior point of set A

$\Leftrightarrow$ A is a nbd of x

$\Leftrightarrow \exists$ an open set G such that $x \in G \subset A$

$\Leftrightarrow x \in \cup \{G \subset A : G \text{ } is \text{ } open\}$

Hence $A^{\circ} = \cup \{G \subset A : G \text{ } is \text{ } open\}$

Theorem - Let (X, T) be a topological space and let $A \subset X$ then

(i) A° is an open set

(ii) A° is the largest open set contained in A

(iii) A is open $\Leftrightarrow A^{\circ} = A$

Proof – (i) $x \in A^{\circ} \Rightarrow x$ is an interior point of set A

$\Rightarrow$ A is a nbd of x

$\Rightarrow \exists$ an open set G such that $x \in G \subset A$

$\Rightarrow$ $G \subset A$ is nbd of each of its point [since G is open]

$\Rightarrow$ A is nbd of each point of G

$\Rightarrow$ Every point of G is an interior point of A

$\Rightarrow G \subset A^{\circ}$

Thus it is shown that to each $x \in A^{\circ}, \exists$ an open set G such that $x \in G \subset A^{\circ}$

$\Rightarrow A^{\circ}$ is a nbd of each of its points.

$\Rightarrow A^{\circ}$ is open.

(ii) Let G be any open subset of A and let $x \in G$

so that $x \in G \subset A$

since G is open, A is a nbd of x ,consequently x is an interior point of A.

i.e. $x \in A^{\circ}$

Thus we have shown that $x \in G \Rightarrow x \in A^{\circ}$

$\therefore$ we have $G \subset A^{\circ}$

Hence A° contains every open subset of A and therefore it is the largest open subset of A.

(iii) let A be open

Then A is open $\Rightarrow A$ itself is the largest open set contained in A

But by (ii), A° is the largest open set contained in A. Hence $A^{\circ} = A$

Conversely let $A^{\circ} = A$

Since by (i), A° is an open set

Hence A is an open set

Properties of interior

Theorem - Let (X, T) be a topological space and let A, B be any subsets of X then

(i) $X^{\circ} = X, \emptyset^{\circ} = \emptyset$

(ii) $A^{\circ} \subset A$

(iii) $A \subset B \Rightarrow A^{\circ} \subset B^{\circ}$

(iv) $A^{\circ} \cup B^{\circ} \subset (A \cup B)^{\circ}$

(v) $(A \cap B)^{\circ} = A^{\circ} \cap B^{\circ}$

(vi) $(A^{\circ})^{\circ} = A$

Proof- (i)$\because X$ is open $\Rightarrow X^{\circ} = X$

Similarly $\emptyset$ is open $\Rightarrow \emptyset^{\circ} = \emptyset$

(ii) $x \in A^{\circ} \Rightarrow x$ is an interior point of set A

$\Rightarrow$ A is a nbd of x

$\Rightarrow x \in A$

Thus $x \in A^{\circ} \Rightarrow x \in A$

Hence $A^{\circ} \subset A$.

(iii) Let $A \subset B$

Then $x \in A^{\circ} \Rightarrow x$ is an interior point of A

$\Rightarrow A$ is a nbd of point x

$\Rightarrow B \supset A$ is a nbd of point x

$\Rightarrow x$ is an interior point of B

$\Rightarrow x \in B^{\circ}$

$\therefore$ we have $A^{\circ} \subset B^{\circ}$

Thus $A \subset B \Rightarrow A^{\circ} \subset B^{\circ}$

(iv) Let A, B be any subsets of X then $A \subset A \cup B$ and $B \subset A \cup B$

Now $A \subset A \cup B \Rightarrow A^{\circ} \subset (A \cup B)^{\circ}$

and $B \subset A \cup B \Rightarrow B^{\circ} \subset (A \cup B)^{\circ}$

hence $A^{\circ} \cup B^{\circ} \subset (A \cup B)^{\circ}$

(v) If A, B are any two subsets of X then

$A \cap B \subset A$ and $A \cap B \subset B$ ....(i)

To prove $(\boldsymbol{A} \cap \boldsymbol{B})^{\circ} \subset \boldsymbol{A}^{\circ} \cap \boldsymbol{B}^{\circ}$ **:** Let $x \in (A \cap B)^{\circ}$ be arbitrary, then

$x \in (A \cap B)^{\circ} \Rightarrow x$ is an interior point of set $A \cap B$

$\Rightarrow \exists$ an open set G such that $x \in G \subset A \cap B$

$\Rightarrow \exists$ an open set G such that $x \in G \subset A$ and $x \in G \subset B$

[Using (i)]

$\Rightarrow x \in A^{\circ}$ and $x \in B^{\circ}$

$\Rightarrow x \in A^{\circ} \cap B^{\circ}$

Thus $x \in (A \cap B)^{\circ} \Rightarrow x \in A^{\circ} \cap B^{\circ}$

$\therefore \qquad (A \cap B)^{\circ} \subset A^{\circ} \cap B^{\circ}$ ....(ii)

To prove $\boldsymbol{A^{\circ} \cap B^{\circ} \subset (A \cap B)^{\circ}}$:

Let $y \in A^{\circ} \cap B^{\circ}$ be arbitrary, then

$y \in A^{\circ} \cap B^{\circ} \Rightarrow \; y \in A^{\circ}$ and $y \in B^{\circ}$

$\Rightarrow \exists$ open sets G_1, G_2 such that $y \in G_1 \subset A$ **and** $y \in G_2 \subset B$

$\Rightarrow \exists$ open set $G_1 \cap G_2$ such that $y \in G_1 \cap G_2 \subset A \cap B$

$\Rightarrow y$ is an interior point of $A \cap B$

$\Rightarrow y \in (A \cap B)^{\circ}$

Thus $y \in A^{\circ} \cap B^{\circ} \Rightarrow \; y \in (A \cap B)^{\circ}$

$\therefore \quad A^{\circ} \cap B^{\circ} \subset (A \cap B)^{\circ}$ ....(iii)

From (ii) and (iii) we have

$$(A \cap B)^{\circ} = A^{\circ} \cap B^{\circ}$$